Magnetic Vector Potential:

Review: In electrostatics we have seen that electric field is conservative

$$\nabla \times \vec{E} = 0,$$

and we can write $\vec{E}$ as negative gradient of a scalar potential (V) $\quad \vec{E} = -\nabla V.$

In previous lecture notes we have seen ~~(in any~~ (in magnetostatics) that

$$\nabla \cdot \vec{B} = 0 \qquad \text{——} \quad (1)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad \text{——} \quad (2)$$

Applying Gauss' and Stokes' theorems in above expressions we obtain

$$\int_V \cancel{a} (\nabla \cdot \vec{B})\, d\vec{x} = \int_S \vec{B} \cdot d\vec{s} = 0, \quad \left\{ \begin{array}{l} \text{using} \\ \text{Gauss'} \\ \text{theorem} \end{array} \right.$$

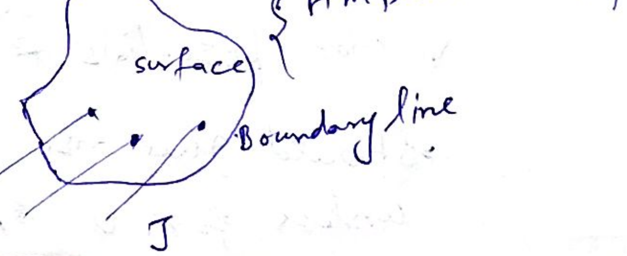
Next, ~~of~~ using Stokes' theorem

$$\int_S (\nabla \times \vec{B}) \cdot d\vec{s} = \oint_C \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{J} \cdot d\vec{s}$$

$$\boxed{\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I} \quad \text{—— (3)} \longrightarrow \text{Ampere's law}$$

$\oint_C \vec{B} \cdot d\vec{l} \longrightarrow$ line integral of magnetic field around a closed loop.

$I = \int_S \vec{J} \cdot d\vec{s} \longrightarrow$ total current passing through the surface



Vector potential:

Generally, in magnetostatics, magnetic field is not conserved. We cannot write $\vec{B}$ in terms of a scalar potential. Since $\nabla\cdot\vec{B}=0$, we can write $\vec{B}$ as curl of a vector field.

From Biot-Savart law we write.

$$\vec{B}=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')\times\vec{r}}{r^3}\,d\vec{x}'$$

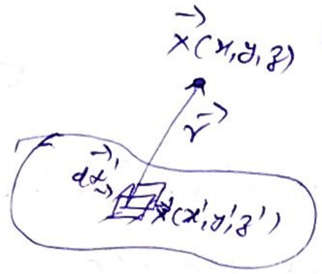
$\vec{x}=(x,y,z)$

$\vec{x}'=(x',y',z')$

$\vec{r}=\vec{x}-\vec{x}'$

$r=|\vec{r}|=|\vec{x}-\vec{x}'|$

$d\vec{x}'=dx'dy'dz'$



$$\text{or}\quad \vec{B}=-\frac{\mu_0}{4\pi}\int d\vec{x}'\,\vec{J}(\vec{x}')\times\nabla\left(\frac{1}{|\vec{r}|}\right)$$

$$\vec{B}=\nabla\times\frac{\mu_0}{4\pi}\int d\vec{x}'\,\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$|\vec{r}|=|\vec{x}-\vec{x}'|$ I have written here for more clarity

Next, we define

$$\vec{B}\equiv\nabla\times\vec{A} \qquad (4)$$

where $\vec{A}$ is called as vector potential.

$$\boxed{\vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d\vec{x}'} \qquad (5)$$

$\nabla\cdot\vec{B}=0$ is also satisfied by eqn (4).

Vector potentials $\vec{A}(\vec{x})$ and $\vec{A}+\nabla\chi(\vec{x})$ should give rise ~~that~~ the same magnetic field, unless for a specified conditions, since curl of a grad is zero. $\lambda(\vec{x})$ introduced here is an arbitrary ~~ϕ~~ scalar function.

Let us calculate divergence of $\vec{A}$

$$\nabla\cdot\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \nabla\cdot\left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] d\vec{x}'$$

since $\vec{J}$ is function of primed varibles only, therefore, ~~grad will~~ where $\nabla\cdot\vec{J}(\vec{x}')$ appears it vanishes.

$$\nabla\cdot\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d\vec{x}'\,\vec{J}(\vec{x}')\cdot\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

where we have used the identity

$$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot\nabla f$$

$$\nabla\cdot\vec{A}(\vec{x}) = -\frac{\mu_0}{4\pi}\int d\vec{x}'\,\vec{J}(\vec{x}')\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$\left\{ \because \frac{\partial}{\partial x} f(x-x') = -\frac{\partial}{\partial x'} f(x-x') \right\}$$

using integration by parts and taking surface term zero we obtain

$$\nabla\cdot\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d\vec{x}'\,\left(\nabla'\cdot\vec{J}(\vec{x}')\right)\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

for steady state current in magnetostatics $\frac{\partial \rho}{\partial t} = 0 \Rightarrow$ $\nabla\cdot\vec{J} = 0$. Therefore we get

$$\boxed{\nabla\cdot\vec{A}(\vec{x}) = 0}$$

H.W. Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.